Problem 1.13

Two infinite grounded parallel conducting planes are separated by a distance d. A point charge q is placed between the planes. Use the reciprocation theorem of Green to prove that the total induced charge on one of the planes is equal to $(-q)$ times the fractional perpendicular distance of the point charge from the other plane. (*Hint*: As your comparison electrostatic problem with the same surfaces choose one whose charge densities and potential are known and simple.)

Solution

The governing equations of the electric field are Gauss's law and Faraday's law. In the context of electrostatics in vacuum they are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = \mathbf{0}.$$

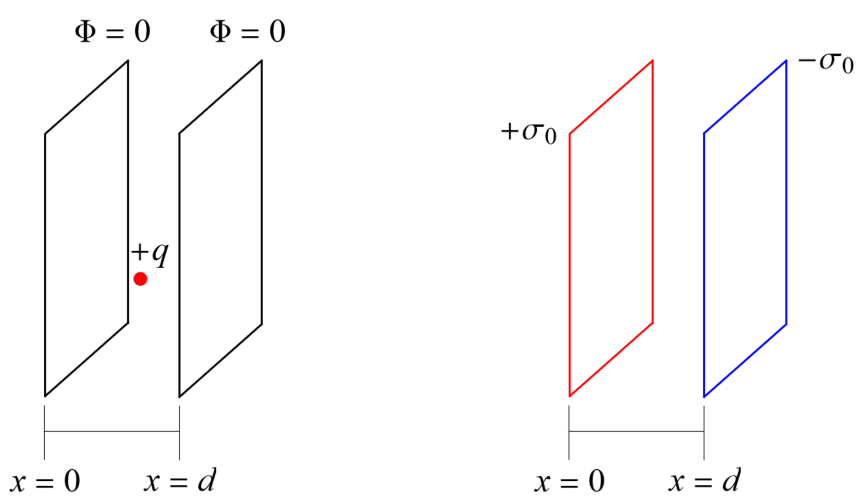
This second equation implies the existence of a potential function $-\Phi$ that satisfies

$$\mathbf{E} = \nabla(-\Phi) = -\nabla\Phi. \tag{1}$$

The minus sign is arbitrary mathematically, but physically it indicates that a positive charge in an electric field moves from high-potential regions to low-potential regions (and vice-versa for a negative charge). Note that Φ is interpreted as the work it takes to move a positive unit charge from the low-potential conductor to the high-potential conductor. According to Green's reciprocation theorem,

$$\iiint_V \rho\Phi'\,dV + \oiint_S \sigma\Phi'\,dS = \iiint_V \rho'\Phi\,dV + \oiint_S \sigma'\Phi\,dS,$$

where Φ is the electrostatic potential due to a volume charge density ρ within a volume V that also has a surface charge density σ on the conducting surface S and Φ' is the potential for the same surface and volume with surface and volume densities, σ' and ρ', respectively.



Here V is the volume sandwiched between the infinite conducting planes, and S consists of the two boundary planes to V. On the left is the situation we're interested in, and on the right is the comparison problem that we can determine the potential of. The electric field between the planes on the right is known to be $\mathbf{E}' = (\sigma_0/\epsilon_0)\hat{\mathbf{x}}$ by the superposition principle. Use equation (1) to determine the electric potential.

$$\mathbf{E}' = -\nabla\Phi' \quad \rightarrow \quad E'_x = -\frac{d\Phi'}{dx} \quad \rightarrow \quad \Phi'(x) = -\int E'_x\,dx$$

$$= -\int \frac{\sigma_0}{\epsilon_0}\,dx$$

$$= -\frac{\sigma_0}{\epsilon_0}x + C$$

Let the potential be zero at the negatively charged plate ($x = d$) to determine C.

$$\Phi'(d) = -\frac{\sigma_0}{\epsilon_0}d + C = 0 \quad \rightarrow \quad C = \frac{\sigma_0}{\epsilon_0}d$$

Consequently,

$$\Phi'(x) = -\frac{\sigma_0}{\epsilon_0}x + \frac{\sigma_0}{\epsilon_0}d$$

$$= \frac{\sigma_0}{\epsilon_0}(d - x).$$

Since there's only empty space between the planes on the right, the charge density ρ' in the volume V is zero. However, on the left there's a charge q between the planes located at (x_0, y_0, z_0), so the charge density ρ in the volume V is $q\delta(x - x_0, y - y_0, z - z_0) = q\delta(\mathbf{x} - \mathbf{x}_0)$. Green's reciprocation theorem becomes

$$\iiint_V \rho\Phi'\,dV + \oiint_S \sigma\Phi'\,dS = \iiint_V \rho'\Phi\,dV + \oiint_S \sigma'\Phi\,dS$$

$$\iiint_V q\delta(\mathbf{x} - \mathbf{x}_0)\frac{\sigma_0}{\epsilon_0}(d - x)\,dV + \oiint_S \sigma_{\text{induced}}\frac{\sigma_0}{\epsilon_0}(d - x)\,dS = \iiint_V (0)\Phi\,dV + \oiint_S \sigma'(0)\,dS$$

$$q\frac{\sigma_0}{\epsilon_0}(d - x_0) + \frac{\sigma_0}{\epsilon_0}\oiint_S \sigma_{\text{induced}}(d - x)\,dS = 0$$

$$q\frac{\sigma_0}{\epsilon_0}(d - x_0) + \frac{\sigma_0}{\epsilon_0}\left[\iint\limits_{x=0} \sigma_{\text{induced}}(d - x)\bigg|_{x=0} dS + \iint\limits_{x=d} \sigma_{\text{induced}}(d - x)\bigg|_{x=d} dS\right] = 0$$

$$q\frac{\sigma_0}{\epsilon_0}(d - x_0) + \frac{\sigma_0}{\epsilon_0}\left[\iint\limits_{x=0} \sigma_{\text{induced}}|_{x=0}\,(d)\,dS + \iint\limits_{x=d} \sigma_{\text{induced}}|_{x=d}\,(0)\,dS\right] = 0$$

$$q\frac{\sigma_0}{\epsilon_0}(d - x_0) + \frac{\sigma_0}{\epsilon_0}\left(d\iint\limits_{x=0} \sigma_{\text{induced}}|_{x=0}\,dS\right) = 0$$

$$q\frac{\sigma_0}{\epsilon_0}(d - x_0) + \frac{\sigma_0}{\epsilon_0}\left(d\,q_{\text{induced}}|_{x=0}\right) = 0.$$

Therefore, solving for the induced charge on the $x = 0$ plane,

$$q_{\text{induced}}|_{x=0} = -q\left(1 - \frac{x_0}{d}\right).$$

The induced charge on the $x = d$ plane can be found by realizing that the total induced charge on both plates must add up to $-q$.

$$q_{\text{induced}}\big|_{x=0} + q_{\text{induced}}\big|_{x=d} = -q$$

$$-q\left(1 - \frac{x_0}{d}\right) + q_{\text{induced}}\big|_{x=d} = -q$$

Therefore, solving for the induced charge on the $x = d$ plane,

$$q_{\text{induced}}\big|_{x=d} = -q\left(\frac{x_0}{d}\right).$$

In either case, the induced charge is $-q$ times the fractional perpendicular distance from the opposite plane.